Universality Principle for Orbital Angular Momentum and Spin in Gravity with Torsion

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Abstract

We argue that compatibility with elementary particle physics requires gravitational theories with torsion to be unable to distinguish between orbital angular momentum and spin. An important consequence of this principle is that spinless particles must move along autoparallel trajectories, not along geodesics.

1. Universality principles provide us with important guidelines for constructing candidates for fundamental theories which have a chance of being true. For example, an essential property of Maxwell's theory is that electromagnetic interactions depend only on the charge of a particle, not on the various physical origins of charge. The charge of an ion is composed of electron and nuclear charges, the latter of proton charges, these in turn of quark charges, which eventually me turn out to arise from further charged substructures. The motion of a charged particle in an electromagnetic field does not depend on these details, which are subject to change by future discoveries. An atom moves like a neutral point particle, in spite of the completely different origins of electron and proton charges, the exact neutrality of an atom being the very basis for the electrostatic stability of large gravitational bodies (and thus for the existence of theoretical physics).

The irrelevance of the physical origin of the "charge" of gravitational interactions, the mass, led Einstein to the discovery of a geometric theory of these interactions. Just as electric charge, also the mass of a particle has

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a multitude of origins, arising from the masses of constituents and various field energies holding these together. Gravitational interactions depend only on the total mass, and this property makes all particles run along the same trajectories, which can therefore be used to define a geometry of spacetime. In Einstein's theory the independence of the physical origin of the mass is ensured by the fact that the Einstein curvature tensor is directly proportional to the total energy momentum tensor of the theory. Its precise composition depends on the actual status of elementary particle physics, but the motion is invariant with respect to this composition, and thus to future discoveries on the internal structure of the particles.

The universality of weak and color charges was an important principle in the construction of unified theories of electromagnetic and weak, as well as of strong interactions.

2. Since a number of years, theoreticians have enjoyed the idea that the geometry of spacetime may not only be curved but carry also torsion. The line of arguments leading to this idea was that Einstein's gravity may be viewed as a gauge theory of local translations. These generalize the global translations under which all local theories are invariant in Minkowski spacetime. But the latter theories are also invariant under the larger Poincaré group, the group of translations and Lorentz transformations. It therefore seemed natural to postulate the existence of a second gauge field which ensures the invariance under local Lorentz transformations [1]. As a result one obtains an interaction between torsion and spin [2] in a four-dimensional spacetime with general coordinates q^μ:

$$\mathcal{A} = -\frac{1}{2}\int d^4q\sqrt{-g}K_{\mu\nu\lambda}\Sigma^{\nu\lambda,\,\mu}, \tag{1}$$

where $K_{\mu\nu\lambda}$ is the contortion tensor, containing the torsion in the combination $K_{\mu\nu\lambda} = S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu}$. The tensor $\Sigma^{\nu\lambda,\,\mu}$ is the local spin current density.

Consider now a particle at rest in a Riemann-flat space with euclidean coordinates $\mathbf{x} = (x^1, x^2, x^3)$ with $x^i = q^i$ for $(i = 1, 2, 3)$, and a time $t = q^0$. We assume the presence of a torsion whose only nonzero components are

$$S_{ij0} = \frac{1}{2}\epsilon_{ijk}b_k, \quad \partial_i b_i = 0, \tag{2}$$

The divergenceless vector $\mathbf{b}$ will at first be assumed to be constant, for sim-

plicity of the argument. Then (1) specifies an interaction energy

$$H_{\rm int} = -\frac{1}{2} b_k \, \frac{1}{2} \int d^3x \, \epsilon_{kij} \Sigma_{ij,0}. \tag{3}$$

For a particle at rest, the factor to the right of b_k is the spin vector S_k of a particle, so that the interaction energy becomes

$$H_{\rm int} = -\frac{1}{2} \mathbf{b} \cdot \mathbf{S}. \tag{4}$$

This looks just like the interaction energy of a unit magnetic moment with a constant magnetic field, and for that reason we shall refer to a torsion of the type (2) as *magneto-torsion*, and the field $\mathbf{b}$ as *torsion-magnetic* field. From the Heisenberg equation $\dot{\mathbf{S}} = i[H, \mathbf{S}]$, we obtain the equation of motion for the spin

$$\frac{d}{dt} \mathbf{S} = -\frac{1}{2} \mathbf{S} \times \mathbf{b}, \tag{5}$$

describing a precession with frequency $\omega = |\mathbf{b}|/2$.

The microscopic origin of the spin of the particle is completely irrelevant for this result. The spin, being the total angular momentum in the particle's rest frame, is composed of the orbital angular momenta of all constituents and their spins. The details of this composition depend on the actual quantum field theoretic description of the particle. A ρ-meson, for instance, has unit spin. From the hadronic strong-interaction viewpoint of bootstrap physics, the unit spin is explained by ρ being a bound state of a pair of spinless pions with unit orbital angular momentum. In quark physics, on the other hand, ρ is a bound state of a quark and an antiquark with zero orbital angular momentum, with spins coupled to unity.

Thus, in the quark description, the spin of a ρ-meson in a torsion field (2) will precess. Clearly, a theory of gravity with torsion can only be consistent with particle physics, if the same precession frequency if found for the hadronic description of ρ as a bound state of two spinless pions.

For present-day theories of gravity with torsion [1], this postulate presents a serious problem. In these theories, the energy momentum tensor $T^{\mu\nu}(q)$ of a spinless point particle satisfies the local conservation law

$$\bar{D}_\nu T^{\mu\nu}(q) = D^*_\nu T^{\mu\nu}(q) + 2S_{\kappa}{}^{\mu}{}_{\lambda}(q) T^{\kappa\lambda}(q) = 0, \quad D^*_\nu \equiv D_\nu + 2S_\nu, \tag{6}$$

where $\bar{D}_\mu$ is the covariant derivative involving the Riemann connection $\bar{\Gamma}_{\mu\nu\lambda}$, and D_μ the covariant derivative involving the full affine connection $\Gamma_{\mu\nu\lambda} = \bar{\Gamma}_{\mu\nu\lambda} + K_{\mu\nu\lambda}$. It is obvious from the torsionless left-hand part of Eq. (6), and was proved in Ref. [3], that such a conservation law leads directly to *geodesic* particle trajectories for point-like spinless particles, governed by the equation of motion

$$\ddot{q}^\nu + \bar{\Gamma}_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa = 0, \tag{7}$$

where $q^\mu(\tau)$ is the orbit parametrized in terms of the proper time τ.

This motion is not influenced by torsion. As a consequence, the spin of a ρ meson at rest would *not* precess in the two-pion description, in contradiction with the quark-antiquark description. Since both descriptions are equally true, we conclude that geodesics cannot be the correct trajectories of spinless particles.

3. The discrepancy can be avoided by another option for the trajectories of spinless particles in this geometry. These are the *autoparallels*, which obey an equation of motion like (7), but with the full affine connection:

$$\ddot{q}^\nu + \Gamma_{\lambda\kappa}{}^\nu \dot{q}^\lambda \dot{q}^\kappa = 0. \tag{8}$$

The conservation law for the energy momentum tensor of a spinless point particle leading to autoparallel motion is [4]

$$D^*_\nu T^{\mu\nu}(q) = 0. \tag{9}$$

In a flat space with torsion, Eq. (8) becomes

$$\ddot{q}^\nu + 2S^\nu{}_{\lambda\kappa} \dot{q}^\lambda \dot{q}^\kappa = 0. \tag{10}$$

Specializing further to a constant magneto-torsion (2), we obtain $\dot{q}^0 = \text{const}$, and find for the spatial motion in euclidean coordinates the equation

$$\frac{d^2}{dt^2}\mathbf{x} = -\dot{\mathbf{x}} \times \mathbf{b}. \tag{11}$$

Thus the constant torsion (2) acts on the orbital motion of the spinless point particle just like a Lorentz force. It is well known from electrodynamics, that this Lorentz force causes a precession of the orbital angular momentum of an

electron. Its frequency is determined by the magnetic moment of the *orbital* motion, whose size for a certain orbital angular momentum $\mathbf{L}$ is half as big as that of a spin $\mathbf{S}$ of equal size. The precession frequency following from (11) is therefore $\omega = |\mathbf{b}|/2$. To show this we simply observe that (11) follows from a Lagrangian $L = \dot{\mathbf{x}}^2/2 + \mathbf{a} \cdot \dot{\mathbf{x}}$, describing a particle of unit mass moving in a *torsion-magnetic* vector potential $\mathbf{a} = \mathbf{b} \times \mathbf{x}$. The associated Hamiltonian depending on $\mathbf{x}$ and the momentum $\mathbf{p} = \dot{\mathbf{x}}$ reads

$$H = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 = \frac{1}{2}\mathbf{p}^2 - \frac{1}{2}\mathbf{b} \cdot (\mathbf{x} \times \mathbf{p}) + \frac{1}{8}(\mathbf{b} \times \mathbf{x})^2. \tag{12}$$

The smallness of the gravitational coupling makes torsion small enough to ignore the last term. From the second term written as $-\frac{1}{2}\mathbf{b} \cdot \mathbf{L}$ calculate from the Heisenberg equation $\dot{\mathbf{L}} = i[H, \mathbf{L}]$ the equation of motion for the orbital angular momentum:

$$\frac{d}{dt}\mathbf{L} = -\frac{1}{2}\mathbf{L} \times \mathbf{b}, \tag{13}$$

which is the same as Eq. (5) for the spin, leading to the same precession frequency $\omega = |\mathbf{b}|/2$.

A similar study can of course by performed for an *electro-torsion* field ${S_{i0}}^0 = e^i/2$ with $e^i = \partial_i a^0$, so that the autoparallel differential equation (10) can be rewritten as

$$\frac{d^2}{dt^2}\mathbf{x} = -\mathbf{e} - \dot{\mathbf{x}} \times \mathbf{b}, \tag{14}$$

thus extending (11) to an analog of the full Lorentz equation. The Hamiltinian (12) contains then an extra electro-torsion term a^0. This Hamiltonian may be quantized straightforwardly to obtain a quantum mechanics in the presence of electromagneto-torsion fields. The eikonal approximation of the Schrödinger wave function will describe autoparallel trajectories.

Although the discussion up to this point assumed constant electromagneto-torsion fields $\mathbf{e}$ and $\mathbf{b}$, it is easy to convince ourselves that the final theory is also valid for space-dependent fields.

Let us compare this with the couplings in proper magnetism, where in analogy to the universal coupling of an electric field to the charge of a particle, a magnetic field $\mathbf{B}$ couples universally to the magnetic moments. For orbital angular momenta and spin, however, the magnetic coupling is nonuniversal. Consider atomic electrons. They have a gyromagnetic ratio $g = 2$ caused

by the Thomas precession, so that the magnetic interaction Hamiltonian is (ignoring the anomalous magnetic moment)

$$H_{\rm int} = -\mu_B \, \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}) \, , \qquad \mu_B \equiv \frac{e}{2Mc} \tag{15}$$

where μ_B is the Bohr magnetic moment (using $\hbar = 1$). In a weak magnetic fields, an atom has an interaction energy $-g\mu_B BM$, with the gyromagnetic ratio $g = 1 + [J(J+1) + S(S+1) - L(L+1)] / 2J(J+1)$, where J is the quantum number of the toral spin vector $\mathbf{J} = \mathbf{L} + \mathbf{S}$. This ratio g causes the characteristic level splitting of the Zeeman effect.

4. It is useful to set up a new action for a Dirac field which is compatible with the proposed universality principle. In a first step, consider a Riemann-flat spacetime with Minkowski coordiantes $x^\alpha = (x^0, \mathbf{x})$ and an action

$$\mathcal{A} = \int d^4x \, \bar{\psi}(x) \left[\gamma^\alpha \left(i f_\alpha{}^\beta \partial_\beta - eA_\alpha - \frac{1}{2} K_{\alpha\beta\gamma} \Sigma^{\beta\gamma} \right) - M \right] \psi(x), \tag{16}$$

where $f_\alpha{}^\beta = 1 - a_\alpha{}^\beta$, with $a^{\alpha 0}$ being the electromagneto-torsion field $(a^0, \mathbf{a})$, and the other components $a_\alpha{}^i = 0$ vanishing. It is a gauge field whose curl yields the torsion, $S_{ij}{}^0 = (\partial_i a_j{}^0 - \partial_j a_i{}^0)/2$. The action is gauge-invariant under $a_i{}^0(x) \to a_i{}^0(x) + \partial_i \Lambda^0(\mathbf{x})$ with a simultaneous transformation $\psi(x) \to e^{-i\Lambda^0(\mathbf{x})\partial_0} \psi(x)$. The 4×4-matrices $\Sigma_{\beta\gamma} \equiv \frac{i}{4}[\gamma_\beta, \gamma_\gamma]_-$ are the generators of Lorentz transformations, so that the spin current density in (1) is $\Sigma_{\beta\gamma,\alpha} = -\frac{i}{2}\bar{\psi}[\gamma_\alpha, \Sigma_{\beta\gamma}]_+ \psi$. Here $[\ .\ ,\ .\ ]_\mp$ denotes commutator and anticommutator, respectively, and all quantities have standard Dirac notation. Now we use the Gordon formula

$$\bar{u}(\mathbf{p}', s_3')\gamma^\alpha u(\mathbf{p}, s_3) = \bar{u}(\mathbf{p}', s_3') \left[\frac{1}{2M}(p'^\alpha + p^\alpha) + \frac{i}{2M}\sigma^{\alpha\beta} q_\beta \right] u(\mathbf{p}, s_3) \tag{17}$$

to calculate between single-electron states of small momenta $\mathbf{p}'$ and $\mathbf{p}$ with momentum transfer $q = p' - p$ the interaction energy for slow electrons

$$\begin{aligned} H_{\rm int} \quad &= \int d^3x \left[\frac{e}{M} \mathbf{A}(x) \cdot (\mathbf{p} {+} \mathbf{q} {-} i\mathbf{q} \times \mathbf{\Sigma}) \right. \\ &\qquad \left. + \ \mathbf{a}(x) \cdot (\mathbf{p} {+} \mathbf{q} {-} i\mathbf{q} \times \mathbf{\Sigma}) - M a^0(x) - \frac{1}{2} \mathbf{b} \cdot \mathbf{\Sigma} \right] e^{-i\mathbf{q}\mathbf{x}}, \end{aligned} \tag{18}$$

where $\Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma_{jk}$ are the Dirac spin matrices. We have omitted the external spinors $\bar{u}(\mathbf{p}', s_3')$ and $u(\mathbf{p}, s_3)$, for brevity, since we shall immediately

take the limit $\mathbf{p}' \to \mathbf{p}$ where $\bar{u}(\mathbf{0}, s_3')u(\mathbf{0}, s_3) = \delta_{s_3' s_3}$, $\bar{u}(\mathbf{0}, s_3')\Sigma_{ij}u(\mathbf{0}, s_3) = \epsilon_{ijk}(S_k)_{s_3' s_3}$, and $S_k = \sigma_k/2$, with Pauli spin matrices σ_k. Before going to this limit, we convert $\mathbf{q}$ into a derivative of $e^{-i\mathbf{qx}}$, then via an integration by parts into a derivative of $\mathbf{A}(x)$, and using the vector potentials $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{x}$ and $\mathbf{a} = \frac{1}{2}\mathbf{b} \times \mathbf{x}$, we obtain in the limit $\mathbf{p}' \to \mathbf{p}$ for a slow electron

$$H_{\rm int} = \int d^3x \left[\alpha_B \, \mathbf{B} \cdot (\mathbf{L} + 2\mathbf{S}) - \frac{1}{2}\mathbf{b} \cdot (\mathbf{L} + \mathbf{S}) - Ma^0 \right]. \tag{19}$$

Observe that the last spin term in (18) has removed precisely half of the spin term coming from the coupling of torsion to γ^i, thus leading to the universal coupling $\mathbf{b} \cdot (\mathbf{L} + \mathbf{S}) = \mathbf{b} \cdot \mathbf{J}$. The Hamiltonian (19) ensures that nonrelativistic electrons follow the equation of motion Eq. (14), thus running along autoparallels (10).

To complete the analogy with magnetism, we make the dimension of the magnetotorsion field equal to that of the magnetic field by defining

$$\mathbf{b} \equiv \alpha_K \, \mathbf{B}^K, \tag{20}$$

with the *torsionmagneton*

$$\alpha_K \equiv \sqrt{G}\hbar/2c, \tag{21}$$

where $G = \hbar c/M_P^2$, and M_P is the Planck mass $M_P = 2.38962 \times 10^{22} \, M$. The torsionmagneton is the same factor smaller than the Bohr magneton.

Note that in present-day gravity with torsion [1, 2], the term $\frac{1}{2}\mathbf{b} \cdot \mathbf{L}$ is absent in (19), while $\frac{1}{2}\mathbf{b} \cdot \mathbf{S}$ is present, in violation of our universlity principle.

5. It is obvious how this theory can be extended to a more general torsion field. We simply allow for a full 4×4 matrix $f_\mu{}^\nu(x)$ in the gradient term of the action (16). Then we allow for a nonvanishing Riemann curvature by introducing a vierbein field $h_\alpha{}^\nu(q)$ which tranforms the Minkowski space locally to general coordinates q^μ with a metric $g_{\mu\nu} = h_{\alpha\mu}h^\alpha{}_\nu$. This adds to the contortion the so-called spin connection $K^h_{\alpha\beta\gamma} = S^h_{\alpha\beta\gamma} - S^h_{\beta\gamma\alpha} + S^h_{\gamma\alpha\beta}$, where $S^h_{\alpha\beta\gamma} \equiv h_\alpha{}^\mu h_\beta{}^\nu \left(\partial_\mu h_{\gamma\nu} - \partial_\nu h_{\gamma\mu}\right)/2$, and arrive at the action

$$\mathcal{A} = \int d^4x \sqrt{-g}\bar{\psi}(x) \left\{ \gamma^\alpha h_\alpha{}^\mu \left[i f_\mu{}^\sigma \partial_\sigma - eA_\mu - \frac{1}{2} A_{\mu\beta\gamma}\Sigma^{\beta\gamma} \right] - M \right\} \psi(x), \tag{22}$$

where $A_{\mu\beta\gamma} \equiv K_{\mu\beta\gamma} + K^h_{\mu\beta\gamma}$, and indices are freely converted between Minkowsi and general coordinates via the matrix $h_\alpha{}^\mu$ and its inverse $h^\alpha{}_\mu$ (satisfying $h_\alpha{}^\mu h^\alpha{}_\nu = \delta^\mu{}_\nu$), and between co- and contravarint via the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. We have also found it convenient to introduce a third group of indices $\sigma, \tau, \omega, \ldots$, so that we may define an inverse $f^\mu{}_\tau$ by $f^\mu{}_\sigma f_\nu{}^\sigma = \delta^\mu{}_\nu$. This action has the usual gauge invariance under local coordinate transformations and local rotations [1, 2]. The nontrivial transformations of the derivative term can be absorbed in the respective gauge fields $h_\alpha{}^\mu$ and $A_{\mu\alpha\beta}$. The covarant curl of $A_{\mu\alpha\beta}$ is the Cartan curvature tensor.

The action (22) satisfies our universality principle. In a Riemann-flat space with electromagnete-torsion, this is ensured by our very construction As another example, consider a space with Riemann curvature and torsion at *zero* Cartan curvature. Then $A_{\mu\alpha\beta} \equiv 0$, i.e., $K_{\mu\beta\gamma} = -K^h_{\mu\beta\gamma}$, or

$$S_{\alpha\beta}{}^\gamma = -h^\gamma{}_\nu \left(h_\alpha{}^\mu \partial_\mu h_\beta{}^\nu - h_\beta{}^\mu \partial_\mu h_\alpha{}^\nu \right)/2 = h_\alpha{}^\mu h_\beta{}^\nu \left(\partial_\mu h^\gamma{}_\nu - \partial_\nu h^\gamma{}_\mu \right)/2. \quad (23)$$

This is a nonlinear version of the same type of electromagneto-torsion as considered above. Since $A_{\mu\alpha\beta} \equiv 0$, there is no direct coupling of the Dirac field to spin. Then, by the universality principle, there should also be no coupling to spin from the derivative term proportional to γ^i. This is indeed true if we define the nonlinear relation between gauge field of torsion $f_\mu{}^\nu$ in the same way as in Eq. (23), except with the matrix h replaced by f^{-1}. Then the torsion in a Cartan-flat space has a gauge field $f = h^{-1}$, so that torsion completely disappears from the Dirac action (22).

6. Note that also in a torsionless spacetime, the universality of orbital angular momentum and spin is satisfied. Then $A_{\mu\alpha\beta} = K^h_{\mu\alpha\beta}$ and $f_\mu{}^\sigma = \delta_\mu{}^\sigma$, and for a nearly flat $h_i{}^0 = \tilde{a}^i$ the gradient term in (22) gives a coupling $\tilde{\mathbf{b}} \cdot (\mathbf{L} + 2\mathbf{S})$, with $\tilde{b}_i = \epsilon_{ijk}\partial_j h_i{}^0$, while the spin term removes half the spin coupling just as in (19), thus leading once more to the universal form $\tilde{\mathbf{b}} \cdot \mathbf{J}$.

Indeed, this can be observed in a well-known result of Einstein's theory of garvitation on the gravitational field of a spinning star which exerts a rotational drag upon a distant point particle (Lense-Thirring effect). The deviation of the metric from the Minkowski form is

$$\begin{aligned} \phi^{00}(\mathbf{x}) &= -4\frac{GM}{c^2 r} + \ldots\ , \qquad \phi^{ji}(\mathbf{x}) = 0, \\ \phi^{0i}(\mathbf{x}) &= \phi^{i0}(\mathbf{x}) = 2\frac{G}{c^3 r^3}(\mathbf{x} \times \mathbf{J}) + \ldots\ , \end{aligned} \quad (24)$$

where G is the gravitational constant, M the mass, and $\mathbf{J}$ the total angular momentum of the star at the origin, obtained from the spatial integral over the star volume V:

$$J^k = \frac{1}{2}\epsilon_{ijk}J^{ij} = \frac{1}{2}\epsilon_{ijk}\int_V d^3x[x^iT^{j0}(\mathbf{x},t) - x^jT^{i0}(\mathbf{x},t)]. \tag{25}$$

The energy-momentum tensor on the right-hand side receives contributions from both orbital as well as spin angular momentum. Thus, a nonrotating polarized neutron star with total spin $\mathbf{S}$ gives rise to the same Lense-Thirring effect as a rotating star composed of spinless dust with purely orbital angular momentum $\mathbf{L}$ equal to the spin $\mathbf{S}$ of the neutron star.

Conversely, a spinning test particle in an external gravitational field $g_{\mu\nu}(q)$ is only coupled via its total energy-momentum tensor $T^{\mu\nu}(q)$. In the rest-frame of a particle, the off-diagonal matrix elements of $T^{\mu\nu}(q)$ receive equal contributions from orbital and spin angular momenta.

7. In conclusion we see that only autoparallel trajectories comply with the universality principle of orbital and spin angular momentum. This principle guarantees our ability to predict the gravitational behavior of particles without detailed knowledge on the source of their spin in terms of its fundamental constituents. In fact, this knowledge will probably never be available completely, since every new generations of physicists discovers additional fundamental particles.

The theory of a Dirac field defined by the action (22) leads to a consistent description of an alectron in a gravitational field in which the torsion is restricted to the to be a curl. It will be useful to study the conservation law for the energy momentum tensor of the electron field following from Eq. (22) and show that the resulting trajectories are autoparallels. The eikonal approximation to the wave propagation should be performed to confirm this.

The present theory is a nontrivial extension of the theory with torsion of the gradient type [4, 5]. The remaining eight torsion components will need further work.

Let us end by remarking that autoparallel equations of motion can be derived from the standard action of a classical point particle action via a modified variational procedure [6, 7, 8] which follows from geometric considerations (closure failure of parallelograms in the presence of torsion). The geometric basis for these developments was derived from an analogy of these

spaces with a crystal with defects, which in crystal play the same geometric role as curvature and torsion in gravity [2]. A *nonholonomic mapping principle* was found [4, 8] to transform equations of motion from flat space to spaces with curvature and torsion. This was a necessary step in solving another fundamental problem, the path integral of the hydrogen atom [8].

Autoparallel trajectoris are also the most natural trajectories obtained from an embedding of spaces with torsion in a Riemannian space [9].

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